

Moments and Absolute Moments of the Normal Distribution

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Abstract

We present formulas for the (raw and central) moments and absolute moments of the normal distribution. We note that these results are not new, yet many textbooks miss out on at least some of them. Hence, we believe that it is worthwhile to collect these formulas and their derivations in these notes.

I. INTRODUCTION

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a normal (Gaussian) random variable (RV) with mean $\mu = E\{X\}$ and variance $\sigma^2 = E\{X^2\} - \mu^2$ (here, $E\{\cdot\}$ denotes expectation). In what follows, we give formulas and derivations for $E\{X^\nu\}$, $E\{(X - \mu)^\nu\}$, $E\{|X|^\nu\}$, and $E\{|X - \mu|^\nu\}$, i.e., for the (raw) moments, the central moments, the (raw) absolute moments, and the central absolute moments. We note that the formulas we present hold for real-valued $\nu > -1$.

The remainder of this text is structured as follows: Section II deals with preliminaries and introduces notation, particularly regarding some special functions. In Section III we present the results and the corresponding derivations are given in Section IV.

II. PRELIMINARIES

We denote the standard deviation by $\sigma = \sqrt{\sigma^2}$. The imaginary unit is $j = \sqrt{-1}$ and z^* denotes the complex conjugate of z . The non-negative integers are denoted by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Next, we give the definitions of subsequently used special functions (cf. [1]).

- *Gamma function:*

$$\Gamma(z) \triangleq \int_0^\infty t^{z-1} e^{-t} dt. \quad (1)$$

- *Rising factorial:*

$$z^{\overline{n}} \triangleq \frac{\Gamma(z+n)}{\Gamma(z)} \quad (2)$$

$$= z(z+1) \cdots (z+n-1), \quad n \in \mathbb{N}_0. \quad (3)$$

- *Double factorial:*

$$z!! \triangleq \sqrt{\frac{2^{z+1}}{\pi}} \Gamma\left(\frac{z}{2} + 1\right) \quad (4)$$

$$= z \cdot (z-2) \cdot \dots \cdot 3 \cdot 1, \quad z \in \mathbb{N} \text{ odd}. \quad (5)$$

- *Kummer's confluent hypergeometric functions:*

$$\Phi(\alpha, \gamma; z) \triangleq M(\alpha, \gamma, z) = {}_1F_1(\alpha; \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha^{\overline{n}} z^n}{\gamma^{\overline{n}} n!}. \quad (6)$$

- *Tricomi's confluent hypergeometric functions:*

$$\Psi(\alpha, \gamma; z) \triangleq U(\alpha, \gamma, z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(\alpha-\gamma+1, 2-\gamma; z). \quad (7)$$

- *Parabolic cylinder functions:*

$$D_\nu(z) \triangleq 2^{\nu/2} e^{-z^2/4} \left[\frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma(-\frac{\nu}{2})} \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right]. \quad (8)$$

III. RESULTS

In this section we give formulas for the raw/central (absolute) moments of a normal RV. If not noted otherwise, these results hold for $\nu > -1$.

- *Raw moments:*

$$\mathbb{E}\{X^\nu\} = (j\sigma)^\nu \exp\left(-\frac{\mu^2}{4\sigma^2}\right) D_\nu\left(-j\frac{\mu}{\sigma}\right) \quad (9)$$

$$= (j\sigma)^\nu 2^{\nu/2} \left[\frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right) + j\frac{\mu}{\sigma} \frac{\sqrt{2\pi}}{\Gamma(-\frac{\nu}{2})} \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}; -\frac{\mu^2}{2\sigma^2}\right) \right] \quad (10)$$

$$= (j\sigma)^\nu 2^{\nu/2} \cdot \begin{cases} \Psi\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right), & \mu \leq 0 \\ \Psi^*\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right), & \mu > 0 \end{cases} \quad (11)$$

$$= \begin{cases} \sigma^\nu 2^{\nu/2} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right), & \nu \in \mathbb{N}_0 \text{ even} \\ \mu \sigma^{\nu-1} 2^{(\nu+1)/2} \frac{\Gamma(\frac{\nu}{2}+1)}{\sqrt{\pi}} \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}; -\frac{\mu^2}{2\sigma^2}\right), & \nu \in \mathbb{N}_0 \text{ odd} \end{cases}. \quad (12)$$

- *Central moments:*

$$\mathbb{E}\{(X-\mu)^\nu\} = (j\sigma)^\nu 2^{\nu/2} \frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})} \quad (13)$$

$$= (1 + (-1)^\nu) \sigma^\nu 2^{\nu/2-1} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}} \quad (14)$$

$$= \begin{cases} \sigma^\nu (\nu-1)!!, & \nu \in \mathbb{N}_0 \text{ even} \\ 0, & \nu \in \mathbb{N}_0 \text{ odd} \end{cases}. \quad (15)$$

- *Raw absolute moments:*

$$\mathbb{E}\{|X|^\nu\} = \sigma^\nu 2^{\nu/2} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right). \quad (16)$$

- *Central absolute moments:*

$$\mathbb{E}\{|X-\mu|^\nu\} = \sigma^\nu 2^{\nu/2} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}}. \quad (17)$$

IV. DERIVATIONS

In this section we give derivations for the previously presented results. Below we use the following two identities which hold for $\gamma \in \mathbb{R}, \nu > -1$, see [2, Sec. 3.462]:

$$\int_{-\infty}^{\infty} (-jx)^\nu e^{-x^2+jx\gamma} dx = \sqrt{2^{-\nu}\pi} e^{-\gamma^2/8} D_\nu\left(\frac{\gamma}{\sqrt{2}}\right), \quad (18)$$

$$\int_0^{\infty} x^\nu e^{-x^2-x\gamma} dx = 2^{-(\nu+1)/2} \Gamma(\nu+1) e^{\gamma^2/8} D_{-\nu-1}\left(\frac{\gamma}{\sqrt{2}}\right). \quad (19)$$

- *Raw moments (9):*

$$\mathbb{E}\{X^\nu\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^\nu \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \quad (20)$$

$$= \sqrt{\frac{2^\nu\sigma^{2\nu}}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} x^\nu \exp\left(-x^2 + x\frac{\mu}{\sigma}\sqrt{2}\right) dx \quad (21)$$

$$\stackrel{(18)}{=} (j\sigma)^\nu \exp\left(-\frac{\mu^2}{4\sigma^2}\right) D_\nu\left(-j\frac{\mu}{\sigma}\right). \quad (22)$$

- *Central moments (13):* Follows directly from (9) with $\Phi(\alpha, \gamma; 0) = 1$ and, hence,

$$D_\nu(0) = 2^{\nu/2} \frac{\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})}. \quad (23)$$

- *Raw absolute moments (16):*

$$\mathbb{E}\{|X|^\nu\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} |x|^\nu \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \quad (24)$$

$$= \sqrt{\frac{2^\nu\sigma^{2\nu}}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \left[\int_0^{\infty} x^\nu \exp\left(-x^2 - x\frac{\mu}{\sigma}\sqrt{2}\right) dx + \int_0^{\infty} x^\nu \exp\left(-x^2 + x\frac{\mu}{\sigma}\sqrt{2}\right) dx \right] \quad (25)$$

$$\stackrel{(19)}{=} \sqrt{\frac{2^\nu\sigma^{2\nu}}{\pi}} \exp\left(-\frac{\mu^2}{4\sigma^2}\right) 2^{-(\nu+1)/2} \Gamma(\nu+1) (D_{-\nu-1}(\mu/\sigma) + D_{-\nu-1}(-\mu/\sigma)) \quad (26)$$

$$= \sqrt{\frac{\sigma^{2\nu}}{2^\nu}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \frac{\Gamma(\nu+1)}{\Gamma(\nu/2+1)} \Phi\left(\frac{\nu+1}{2}, \frac{1}{2}; \frac{\mu^2}{2\sigma^2}\right) \quad (27)$$

$$= \sqrt{\frac{2^\nu\sigma^{2\nu}}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \Gamma\left(\frac{\nu+1}{2}\right) \Phi\left(\frac{\nu+1}{2}, \frac{1}{2}; \frac{\mu^2}{2\sigma^2}\right) \quad (28)$$

$$= \sigma^\nu 2^{\nu/2} \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}} \Phi\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right), \quad (29)$$

where we have used Kummer's transformation [2, Sec. 9.212], i.e.,

$$\Phi(\alpha, \gamma; z) = e^z \Phi(\gamma - \alpha, \gamma; -z), \quad (30)$$

in the last step.

- *Central absolute moments (17):* Follows directly from (16) with $\Phi(\alpha, \gamma; 0) = 1$.

REFERENCES

- [1] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., B. Eckmann and B. van der Waerden, Eds. Springer-Verlag, 1966.
- [2] I. Gradshteyn and I. Ryzhik, *Table of integrals, series, and products*, 6th ed., A. Jeffrey and D. Zwillinger, Eds. Academic Press, July 2000.